The quantum harmonic oscillator on a circle – fragmentation of the algebraic method

Daniel Burgarth[1], **Paolo Facchi**[2]

[1] *Department Physik, FAU Erlangen-Nürnberg, Staudtstraße 7, 91058 Erlangen, Germany*
[2] *Dipartimento di Fisica, Università di Bari, I-70126 Bari, Italy*

The harmonic oscillator is the workhorse of quantum optics. In suitable units $\omega = m = \hbar = 1$ its Hamiltonian is given by $H = \frac{1}{2}\left(p^2 + q^2\right)$ where q and p are the canonical position and momentum operators respectively. Using creation and annihilation operators $a^\dagger = (q - \mathrm{i}p)/\sqrt{2}$, $a = (q + \mathrm{i}p)/\sqrt{2}$ the Hamiltonian is equivalently written in normal and antinormal ordered form $H_N = a^\dagger a + \frac{1}{2}$, $H_A = aa^\dagger - \frac{1}{2}$. Using the algebraic ladder method, it is then easily seen that its spectrum is equidistant $\mathbb{N} + \frac{1}{2}$.

In the last decades these methods from quantum optics have made their way into superconducting electrical circuits, central to quantum technology and leading to the 2025 Nobel prize for the discovery of energy quantisation in an electric circuit. There is however an active debate if the charge and flux degrees of freedom in superconductors, which fulfill canonical commutation relations and thereby play the role of p and q respectively, are living on a full line or on a circle [**Devoret21**]. It raises the question if the algebraic method survives on the circle?

We will show [**Facchi26**] that on the circle, the algebraic method fragments: the spectra of H, H_N and H_A are no longer equal and are not equidistant ('harmonic'). In fact, H_A even has a negative eigenvalue $-\frac{1}{2}$. Some levels remain equidistant (Fig. 1).

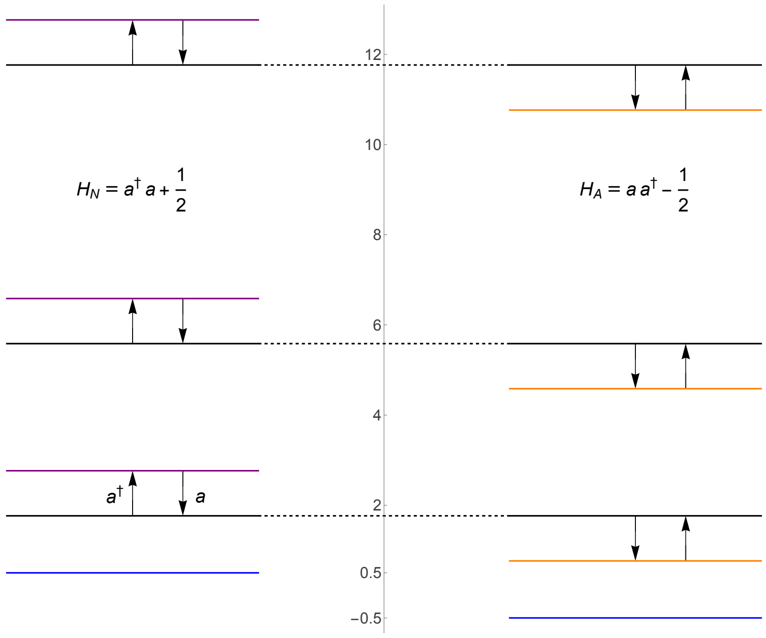

Figure 1: The lowest energy levels of H_N and H_A on a circle of circumference 4. Shown are the remnants of the usual ladder structure: starting on odd levels (black), we obtain new eigenstates of H_N (purple) by applying $a^\dagger$ and of H_A (orange) by applying a. This works only once for each odd level.

These results show that foundational algebraic tools of quantum optics can break down in compact configuration spaces, with potential consequences for oscillator-based descriptions of superconducting quantum circuits.

References

[1] M. H. Devoret, *Does Brian Josephson's gauge-invariant phase difference live on a line or a circle?* J. Supercond. Nov. Magn. **34**, 1633 (2021).

[2] D. Burgarth and P. Facchi, *The quantum harmonic oscillator on a circle – fragmentation of the algebraic method*, arXiv:2603.23774.